

Exact Wavefunctions in a Noncommutative Field Theory*

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We consider the nonrelativistic field theory with a quartic interaction on a noncommutative plane. We compute the $2 \rightarrow 2$ scattering amplitude within perturbative analysis to all orders and identify the beta function and the running of the coupling constant. Since the theory admits an equivalent description via the N particle Schrödinger equation, we regain the scattering amplitude by finding an exact scattering wavefunction of the two body equation. The wave function for the bound state is also identified. These wave functions unusually have two center positions in the relative coordinates. The separation of the centers is in the transverse direction of the total momentum and grows linearly with the noncommutativity scale and the total momentum, exhibiting the stringy nature of the noncommutative field theory.

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As shown recently, quantum field theories in noncommutative spacetime naturally arise as a decoupling limit of the worldvolume dynamics of D-branes in a constant NS-NS two form background[1]. The dynamical effects of the noncommutative geometries mainly in the classical context have been investigated[1, 2, 3, 4, 5, 6]. The quantum aspects of the noncommutative field theories are also pursued via perturbative analysis over diverse models[7, 8, 9, 10], but the understanding is still partial.

In this note, we shall consider the nonrelativistic system with a contact interaction[11, 12] on a noncommutative plane. Performing the perturbative analysis, we obtain the $2 \rightarrow 2$ scattering amplitude exactly. We prove that the theory is renormalizable in the two particle sector and identify the running of the coupling constant to all orders. The theory admits an equivalent description via the N-particle Schrödinger equation. We find the scattering and the bound state wavefunctions exactly and show that the bound state energy and the scattering amplitude agree with those from the perturbation theory. It is observed that the exact two particle wavefunctions have two centers, whose separation is in the transverse direction of the total momentum. The separation grows linearly as the noncommutative scale and the magnitude of the total momentum, exhibiting the stringy nature of the noncommutative field theory[13].

We shall consider a nonrelativistic scalar field theory on a noncommutative plane described by the Lagrangian,

$$L = \int d^2x \left(i \psi^\dagger \partial_t \psi + \frac{1}{2} \psi^\dagger \nabla^2 \psi - \frac{v}{4} \psi^\dagger * \psi^\dagger * \psi * \psi \right), \quad (1)$$

where the $*$ -product (Moyal product) is defined by

$$a(\mathbf{x}) * b(\mathbf{x}) \equiv \left(e^{\frac{i}{2} \theta^{ij} \partial_i \partial'_j} a(\mathbf{x}) b(\mathbf{x}') \right) \Big|_{\mathbf{x}=\mathbf{x}'} \quad (2)$$

with $\theta^{ij} (= \theta \epsilon^{ij})$ being antisymmetric. Classically, the system in the ordinary spacetime possesses the scale invariance as well as the familiar Galileo symmetry. The scale invariance is broken quantum mechanically due to short distance singularities of the contact interaction, producing anomalies. In the noncommutative case, the full Galileo invariance is lost but the rotational and translational symmetries remain because the Moyal product is not covariant under only the boost operations. As a consequence, relative degrees of motion do not in general decouple from total translational motion of the system. But the energy and momentum tensor is defined because the time and spatial translations are symmetries of the system. The breaking of the scale invariance in the noncommutative case is two fold. It is from both the quantum effect and the explicit scale dependence of the Moyal product. The global $U(1)$ invariance under $\psi \rightarrow e^{i\alpha} \psi$ persists in the noncommutative case and the number operator $N = \int d^2x \psi^\dagger \psi$ is still conserved.

We shall quantize the system not by the path integral approach but by the canonical quantization methods imposing the canonical commutation relation

$$[\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}'). \quad (3)$$

The Hamiltonian is given by

$$H = H_0 + V = \int d^2x \left(-\frac{1}{2} \psi^\dagger \nabla^2 \psi + \frac{v}{4} \psi^\dagger * \psi^\dagger * \psi * \psi \right) \quad (4)$$

where H_0 and V denote, respectively, the free Hamiltonian and the contact interaction term. The field in the Schrödinger picture can be expanded in its Fourier components as

$$\psi(\mathbf{x}) = \int \frac{d^2k}{(2\pi)^2} a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (5)$$

and we shall define a vacuum state $|0\rangle$ by $a(\mathbf{k})|0\rangle = 0$. Then the full two point Green function $\langle 0|T(\psi(\mathbf{x}, t)\psi^\dagger(\mathbf{x}', t'))|0\rangle$ can be computed as follows. First notice that $\psi(\mathbf{x}, t) = e^{iHt}\psi(\mathbf{x})e^{-iHt}$ and $e^{-iHt} = e^{-iH_0t}T(\exp[-i\int_0^t dt' V_I(t')])$, where interaction picture operators are defined by $O_I(t) = e^{iH_0t}Oe^{-iH_0t}$. Further using the facts $\langle 0|[\psi_I(\mathbf{x}, t), V_I(t_1)\cdots V_I(t_n)] = 0$ and $\langle 0|V_I(t) = 0$, one finds that $\langle 0|T(\psi(\mathbf{x}, t)\psi^\dagger(\mathbf{x}', t'))|0\rangle = \langle 0|T(\psi_I(\mathbf{x}, t)\psi_I^\dagger(\mathbf{x}', t'))|0\rangle$. Namely, the full two point Green function is the same as that of the free Schrödinger field. The expression for the full two point Green function reads explicitly

$$\langle 0|T(\psi(\mathbf{x}, t)\psi^\dagger(\mathbf{x}', t'))|0\rangle = \int \frac{d^2k d\omega}{(2\pi)^3} \frac{i}{\omega - \frac{k^2}{2} + i\epsilon} e^{-i\omega(t-t') + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \quad (6)$$

This implies that the full propagator is not corrected perturbatively, for example, by the tadpole Feynman diagram. In other words, the tadpole diagram is absent in the perturbative scheme defined below[†].

Perturbation Theory

Our first goal is to calculate the $2 \rightarrow 2$ scattering amplitude within perturbative analysis and to compare this with the result from two particle Schrödinger equation. We shall adopt the perturbation theory defined by the canonical quantization methods. The propagator of the bosonic field is given by

$$D(k_0, \mathbf{k}) = \frac{i}{k_0 - \frac{k^2}{2} + i\epsilon}. \quad (7)$$

The denominator is linear in energy, resulting in propagation that is only forward in time. For this reason, the number of nonvanishing Feynman diagrams is reduced a lot compared to the relativistic version of the theory[12]. The identification of the interaction vertex

$$\Gamma_0\left(\mathbf{k}_1 + \mathbf{k}_2, \frac{\mathbf{k}_1 - \mathbf{k}_2}{2}; \mathbf{k}_3 + \mathbf{k}_4, \frac{\mathbf{k}_3 - \mathbf{k}_4}{2}\right) = -iv_0 \cos(k_1 \wedge k_2) \cos(k_3 \wedge k_4), \quad (8)$$

is straightforward where v_0 denotes the bare coupling constant and $k \wedge k' \equiv \frac{\theta}{2}\epsilon^{ij}k_i k'_j$. This also agrees with the expression in Ref. [9]. The propagator and vertex are presented diagrammatically in Fig. 1. At this level, setting θ to zero, one should recover the corresponding propagator and vertex of the theory in the ordinary plane.

For the scattering amplitude, we compute the on shell four point function. The one loop bubble Feynman diagram is described in Fig. 2 and all the 1PI Feynman diagrams for the four point function are depicted. Apart from an energy and momentum conserving delta function, the one loop bubble diagram is obtained evaluating the following expression,

$$\Gamma_b = \frac{1}{2} \int d^2q dq_0 \Gamma_0(\mathbf{P}, \mathbf{k}; \mathbf{P}, \mathbf{q}) \Gamma_0(\mathbf{P}, \mathbf{q}; \mathbf{P}, \mathbf{k}') D\left(\frac{E_t}{2} + q_0, \frac{\mathbf{P}}{2} + \mathbf{q}\right) D\left(\frac{E_t}{2} - q_0, \frac{\mathbf{P}}{2} - \mathbf{q}\right) \quad (9)$$

[†]The full Green function here is hence different from that in Ref. [9] where the authors used the path integral quantization methods, in which the ordering prescribed in the Hamiltonian (4) is not implemented.



Figure 1: The Feynman diagrams for the propagator and the vertex.

where E_t denotes the total energy and the factor a half in front of the integral comes from the symmetry consideration. Once q_0 integration is performed, we are left with the UV divergent integral with respect to \mathbf{q} . This will be regulated by introducing a large momentum cut-off Λ . The

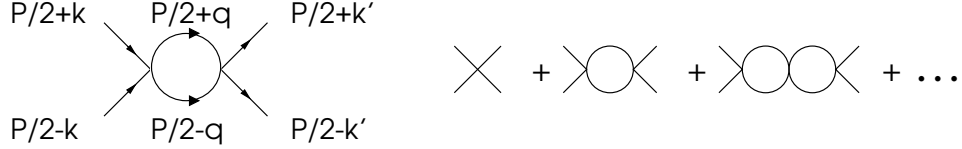


Figure 2: The bubble diagram and all the 1PI diagrams for the four point function.

contribution of the bubble diagram then reads[9],

$$\begin{aligned}\Gamma_b &= \frac{iv_0^2}{8\pi} \cos(k \wedge P) \cos(k' \wedge P) \int \frac{d^2 q}{2\pi} \frac{1 + \cos(2q \wedge P)}{q^2 - k^2 - i\epsilon} \\ &= \frac{iv_0^2}{8\pi} \cos(k \wedge P) \cos(k' \wedge P) [Z(\Lambda/k) + K_0(-i\theta k P)]\end{aligned}\quad (10)$$

where $K_0(x)$ is the Bessel function of the imaginary argument and $Z(x) \equiv \ln x + \frac{i\pi}{2}$. If one set θ to zero before sending the cut-off Λ large, one would recover the one loop amplitude of the ordinary field theory, in which $Z + K_0$ above is replaced by $2Z(\Lambda/k)$. Taking instead the $\theta \rightarrow 0$ limit of (10), the amplitude of the ordinary field theory is not obtained, but the expression becomes singular. In other words, the $\theta \rightarrow 0$ limit does not commute with the $\Lambda \rightarrow \infty$ limit. Since it is fairly clear how to treat the case $\theta P = 0$, we shall below restrict our considerations to the case where θP is nonvanishing. The double bubble, the triple bubble diagrams and so on, can also be computed similarly. Summing these to all loops, one obtains the full four point function as

$$\Gamma = \frac{-i \cos(k \wedge P) \cos(k' \wedge P)}{\frac{1}{v_0} + \frac{1}{8\pi} [Z(\Lambda/k) + K_0(-i\theta k P)]}\quad (11)$$

A few comments are in order. For small x , $K_0(-ix)$ can be expanded as $K_0(-ix) = Z(2/(\gamma x)) + O(x^2)$ with γ being the Euler's constant. Hence, one may see that the UV scale Λ and the IR scale θP combine in the four point function, replacing $Z(\Lambda/k) + K_0(-i\theta k P)$ by $2Z((2\Lambda)^{\frac{1}{2}}/(\gamma\theta P k^2)^{\frac{1}{2}})$ for small θP . The renormalization is achieved by redefining coupling constant v_0 ,

$$\frac{1}{v(\mu)} = \frac{1}{v_0} + \frac{1}{8\pi} \ln\left(\frac{\Lambda}{\mu}\right)\quad (12)$$

where μ is an arbitrary renormalization scale[§]. Alternatively, the renormalization can be achieved by adding a counterterm, $c_0 V$, to the Lagrangian order by order. This proves that the theory is renormalizable in the two particle sector. The exact beta function controlling the running of the

[§]For the consistency of this relation, the bare coupling constant should be negative[12]

coupling constant is found to be $\beta(v(\mu)) \equiv \mu \frac{\partial v(\mu)}{\partial \mu} = \frac{v^2(\mu)}{8\pi}$, which is a half of the beta function arising in the corresponding ordinary field theory. In terms of the the renormalized coupling constant, the $2 \rightarrow 2$ scattering amplitude is

$$A(\mathbf{k}, \mathbf{k}'; \mathbf{P}) = -\frac{1}{4\sqrt{\pi k}} \frac{\cos(k \wedge P) \cos(k' \wedge P)}{\frac{1}{v(\mu)} + \frac{1}{8\pi}[Z(\mu/k) + K_0(-i\theta k P)]} \quad (13)$$

where an appropriate kinematical factor is included. Later, we shall compare this with the result from two body Schrödinger equation. As said earlier, the amplitude depends upon the total momentum P signalling that the boost operation is no longer a symmetry of the system. The imaginary poles of the scattering amplitude indicate that there are bound states. The bound state energy $E_B = -\epsilon_B$ ($\epsilon_B \geq 0$) is obtained by solving $1 = -\frac{v(\mu)}{8\pi}[Z(-i\mu/\sqrt{\epsilon_B}) + K_0(\theta\sqrt{\epsilon_B}P)]$ and there is a single bound state for given $v(\mu)$ since $Z+K_0$ is a monotonic function of ϵ_B covering the range $(-\infty, \infty)$. There are also poles with real k that correspond to resonant states. These states were absent in the case of the ordinary field theory. Further details will be investigated later in the quantum mechanical setting.

Two Particle Schrödinger Equation

As mentioned before, the number operator is conserved in the Schrödinger system. The eigenvalue of the number operators can be shown to be a nonnegative integer that counts particle number. One may then construct two particle Schrödinger equation by the following manner. First note that the operator Schrödinger equation is given by

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(\mathbf{r}) &= [\psi(\mathbf{r}), H] \\ &= -\frac{1}{2}\nabla^2\psi(\mathbf{r}) + \frac{v}{4}\int d^2x \left(\delta(\mathbf{r}-\mathbf{x}) * \psi^\dagger(\mathbf{x}) + \psi^\dagger(\mathbf{x}) * \delta(\mathbf{r}-\mathbf{x}) \right) * \psi(\mathbf{x}) * \psi(\mathbf{x}) \end{aligned} \quad (14)$$

where the time argument of the Schrödinger field operator is suppressed for simplicity. The two particle wavefunction may be constructed by projecting a generic state $|\Phi\rangle$ to two particle sector, i.e. $\phi(\mathbf{r}, \mathbf{r}') = \langle 0 | \psi(\mathbf{r}, t) \psi(\mathbf{r}', t) | \Phi \rangle$. Using the operator Schrödinger equation and evaluating $i\dot{\phi}(\mathbf{r}, \mathbf{r}')$, one is led to the two particle Schrödinger equation,

$$i\dot{\phi}(\mathbf{r}, \mathbf{r}') = -\frac{1}{2}(\nabla^2 + \nabla'^2)\phi(\mathbf{r}, \mathbf{r}') + \frac{v}{4}\int d^2x [\delta(\mathbf{r}-\mathbf{x}) * \delta(\mathbf{r}'-\mathbf{x}) + \delta(\mathbf{r}'-\mathbf{x}) * \delta(\mathbf{r}-\mathbf{x})] * \phi_*(\mathbf{x}, \mathbf{x}) \quad (15)$$

where $\phi_*(\mathbf{x}, \mathbf{x}') = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j}\phi(\mathbf{x}, \mathbf{x}')$. For the momentum space representation, we define $\phi(\mathbf{Q}, \mathbf{q}) = \int d^2r d^2r' e^{-\frac{i}{2}\mathbf{Q}\cdot(\mathbf{r}+\mathbf{r}') - i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} \phi(\mathbf{r}, \mathbf{r}')$. The equation in the momentum space is then given by

$$i\dot{\phi}(\mathbf{Q}, \mathbf{q}) = \left(\frac{1}{4}Q^2 + q^2 \right) \phi(\mathbf{Q}, \mathbf{q}) + \frac{v_0 \cos(q \wedge Q)}{8\pi^2} \int d^2q' \cos(q' \wedge Q) \phi(\mathbf{Q}, \mathbf{q}'). \quad (16)$$

Setting $\phi(\mathbf{Q}, \mathbf{q}) = \delta(\mathbf{Q}-\mathbf{P})\varphi(\mathbf{P}, \mathbf{q})e^{-i(\frac{1}{4}P^2+E_r)t}$, the Schrödinger equation is reduced to

$$(E_r - q^2)\varphi(\mathbf{P}, \mathbf{q}) = \frac{v_0 \cos(q \wedge P)}{8\pi^2} \int d^2q' \cos(q' \wedge P) \varphi(\mathbf{P}, \mathbf{q}'). \quad (17)$$

For the bound state with $E_r = -\epsilon_B$, the equation takes a form

$$\varphi(\mathbf{P}, \mathbf{q}) = \frac{\cos(q \wedge P)}{q^2 + \epsilon_B} C(\mathbf{P}). \quad (18)$$

with $C(\mathbf{P}) = -\frac{v_0}{8\pi^2} \int d^2q' \cos(q' \wedge P) \varphi(\mathbf{P}, \mathbf{q}')$. Integrating the both sides for \mathbf{q} with the weighting factor $\cos(q \wedge P)$, and regulating the resulting integral by the momentum cut off Λ , give an eigenvalue equation for the energy

$$1 = -\frac{v_0}{8\pi} [Z(-i\Lambda/\sqrt{\epsilon_B}) + K_0(\theta P \sqrt{\epsilon_B})] . \quad (19)$$

This condition is precisely the one obtained in the perturbation theory and the renormalization may be achieved by the same way as in the perturbative analysis. The bound state energy is determined uniquely as a function of the renormalized coupling constant, the renormalization scale μ and the external momentum. For small θP , it can be explicitly solved by

$$E_r = -\left(\frac{2\mu}{\gamma\theta P}\right)^{\frac{1}{2}} e^{\frac{8\pi}{v(\mu)}} . \quad (20)$$

From (18), one may get the explicit form of the position space wavefunction,

$$\phi = e^{i\mathbf{P} \cdot \mathbf{R}} \left[K_0 \left(\sqrt{\epsilon_B} |\mathbf{x} - \frac{1}{2}\theta\tilde{\mathbf{P}}| \right) + K_0 \left(\sqrt{\epsilon_B} |\mathbf{x} + \frac{1}{2}\theta\tilde{\mathbf{P}}| \right) \right] , \quad (21)$$

where $\tilde{P}^i = \epsilon^{ij} P_j$, $\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}$ and the relative position \mathbf{x} denotes $\mathbf{r}_1 - \mathbf{r}_2$. The wavefunction has two distinguished centers at $\pm \frac{1}{2}\theta\tilde{\mathbf{P}}$ in the relative coordinates. The separation of these centers is transverse to the direction of the total momentum P . Furthermore, the separation grows linearly in the noncommutativity scale as well as the total momentum. This behavior is precisely the one expected in the open string theory in the constant NS-NS two form background[13], signalling the stringy nature of the field theories in noncommutative spacetime.

We now turn to the scattering problem. In the time independent momentum space equation in (17), the scattering solution can be found with $E_r = k^2$ with an initial relative momentum \mathbf{k} . The exact scattering solution takes a form

$$\varphi(\mathbf{P}, \mathbf{q}) = (2\pi)^2 \delta(\mathbf{q} - \mathbf{k}) + C(\mathbf{P}) \frac{\cos(q \wedge P)}{q^2 - k^2 - i\epsilon} , \quad (22)$$

where $C(\mathbf{P})$ is as given above and $-i\epsilon$ is added to select out the retarded Green function. Since $C(\mathbf{P})$ is dependent upon φ , it should be fixed self consistently. We again integrate the both sides of the above equation with respect to \mathbf{q} with weighting factor $\cos(q \wedge P)$ and obtain

$$C(\mathbf{P}) = -\frac{1}{2} \frac{\cos(k \wedge P)}{\frac{1}{v_0} + \frac{1}{8\pi} [Z(\Lambda/k) + K_0(-i\theta k P)]} . \quad (23)$$

The exact scattering wavefunction is then obtained from (22) as

$$\varphi(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} + \frac{i}{8} \left[H_0^{(1)} \left(k |\mathbf{r} - \frac{1}{2}\theta\tilde{\mathbf{P}}| \right) + H_0^{(1)} \left(k |\mathbf{r} + \frac{1}{2}\theta\tilde{\mathbf{P}}| \right) \right] C(\mathbf{P}) . \quad (24)$$

where $H_\nu^{(1)}(x)$ is the Hankel function of the first kind. The wave function takes a form in the asymptotic region as

$$\varphi(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} + \frac{1}{\sqrt{r}} f(\alpha) e^{i(kr + \frac{\pi}{4})} . \quad (25)$$

with the scattering amplitude

$$f(\alpha) = -\frac{1}{4\sqrt{\pi k}} \frac{\cos(k \wedge P) \cos(k' \wedge P)}{\frac{1}{v_0} + \frac{1}{8\pi}[Z(\Lambda/k) + K_0(-i\theta k P)]}, \quad (26)$$

where $k'_i = k\hat{r}_i$ and α is the scattering angle between the initial momentum \mathbf{k} and the observation direction $\hat{\mathbf{r}}$ [¶]. The renormalization is again completed redefining the coupling constant as (12) while replacing Λ by μ , and the scattering amplitude perfectly agrees with the field theoretic result. The wave function has again two centers, whose separation grows linearly with θP to the transverse direction of the total momentum. We note that the second cosine factor in the scattering amplitude originates from the separation of the two centers and hence this peculiar behavior is even essential in obtaining the correct scattering amplitude.

In this note, we have verified that the theory is renormalizable in the two particle sectors. In the 1PI $n(\geq 6)$ -point Green function, the relevant loop integrals are UV finite. Thus the theory is expected to be renormalizable for any sectors. We leave the details for future works. Finally, we remark that our model may be relevant to studying the light-cone description of four dimensional relativistic field theory with $\theta^{\mu\nu}$ in the transverse directions only. This is because the model is reduced to a nonrelativistic theory on a noncommutative plane though the interactions are typically more general than the one considered here.

There are many directions to go further. If one considers a fermionic nonrelativistic field on an ordinary plane, the contact interaction term vanishes identically due to the anticommuting nature of the fermionic field. On the noncommutative plane, the fermionic contact interaction becomes nontrivial and clarifications are required especially for the nature of the wavefunctions. Rather trivial generalization of the system will include n component generalization or matrix Schrödinger field, and the role of the contact interactions will also be of interest. Another is to couple gauge fields to the noncommutative system. One of the simple option for this direction is to couple the Chern-Simons gauge field, which describes Aharonov-Bohm interaction between particles in case of the ordinary field theory[14]. The effect of the noncommutativity on the Aharonov-Bohm interaction requires further studies.

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[¶]Since we are dealing with identical particles, the wave function in (24) should be symmetrized under $\mathbf{r} \rightarrow -\mathbf{r}$. This effect is added to (26) by the factor $\sqrt{2}$.

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